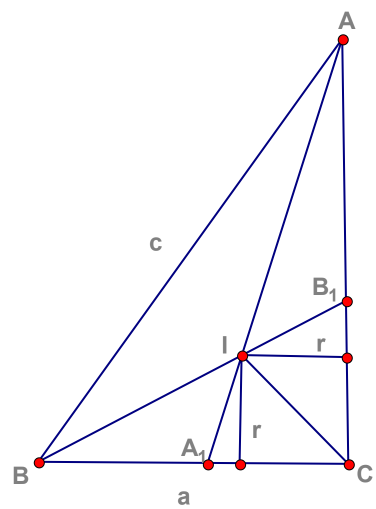
Maximum area of bisectorial quadrilateral in right triangle.

Problem with a solution proposed by Arkady Alt , San Jose , California, USA.

Let F be area of a right triangle $\triangle ABC$ with right angle in vertex C and let AA_1, BB_1 be bisectors of acute angles $\angle A$ and $\angle B$, respectively. Find the right triangle with greatest value of area of "bisectoria" quadrilateral A_1CB_1I .



Solution.

Let r be inradius of $\triangle ABC$ and let $F := \dfrac{[A_1CB_1I]}{[ABC]}$. Since $[ABC] = \dfrac{ab}{2}$, $r = \dfrac{a+b-c}{2}$ and $A_1C = \dfrac{ab}{b+c}, B_1C = \dfrac{ab}{a+c}$ then $[A_1CB_1I] = [A_1CI] + [B_1CI] =$ $\dfrac{r}{2} \cdot A_1C + \dfrac{r}{2} \cdot B_1C = \dfrac{rab}{2}\left(\dfrac{1}{b+c} + \dfrac{1}{a+c}\right) = \dfrac{[ABC](a+b-c)(a+b+2c)}{2(a+c)(b+c)} \Leftrightarrow$

$$F = \frac{(a+b-c)(a+b+2c)}{2(a+c)(b+c)}.$$

Due to homogeneity of $\dfrac{(a+b-c)(2c+a+b)}{2(a+c)(b+c)}$ we can assume that $c = \sqrt{a^2+b^2} = 1$.

Let $t := a+b \leq \sqrt{2(a^2+b^2)} = \sqrt{2}$. Then $2(a+c)(b+c) = 2ab + a^2 + b^2 + 2c(a+b) + c^2 =$ $(t+1)^2$, $\dfrac{(a+b-c)(2c+a+b)}{2(a+c)(b+c)} = \dfrac{(t-1)(t+2)}{(t+1)^2}$ and,

therefore, $\max F = \max \dfrac{(t-1)(t+2)}{(t+1)^2}$.

Since $t+1 \leq \sqrt{2}+1 \Leftrightarrow \dfrac{1}{t+1} \geq \sqrt{2}-1$ we obtain $\dfrac{(t-1)(t+2)}{(t+1)^2} = \dfrac{t^2+t-2}{(t+1)^2} =$ $1 - \dfrac{1}{t+1} - \dfrac{2}{(t+1)^2} \leq 1 - \dfrac{1}{t+1} - \dfrac{2}{(t+1)^2} \leq 1 - \left(\sqrt{2}-1\right) - 2\left(\sqrt{2}-1\right)^2 = 3\sqrt{2}-4$,

where equality occurs iff $t = \sqrt{2} \Leftrightarrow a = b$. Thus, $\max F = 3\sqrt{2}-4$ and is attained iff $a = b$.